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ANTIPLANE SHEAR OF A DOMAIN WITH TWO CLOSELY LOCATED CRACKS*

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Antiplane deformation of a domain with two parallel cracks of different lengths, where the smaller can be above the larger or displaced relative to it, is investigated. A method of solving such problems was proposed in /1/. The spacing between the cracks is considered to be a small parameter of the problem. An approximate solution (the deplanation asymptote) is constructed and, consequently, asymptotic formulas are sought for the stress intensity factors. Crack interaction was investigated numerically in /2, 3/.

1. Formulation of the problem. Let Ω be a domain in the plane $\mathbf{R}^2$ with a smooth (in the class C^∞) boundary $\partial\Omega$, containing the segment $N=\{x\in\mathbf{R}^2:\ x_2=0,\ -1\leqslant x_1\leqslant 1\}$. We introduce still another segment dependent on the small positive parameter ε $M_\varepsilon=\{x:\ x_2=\varepsilon,\ a\leqslant x_1\leqslant b\}$ and the domain $\Omega_\varepsilon=\Omega\setminus(N\cup M_\varepsilon)$ (Fig.1). Here a and b are numbers in the interval $(-1,1)$; $\varepsilon\ll\min\{1-b, 1+a, a+b\}$. We will examine the antiplane shear problem in the domain Ω_ε

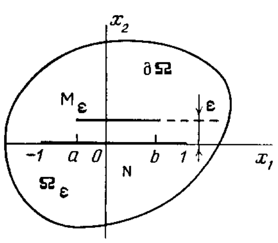

Fig.1

$$\mu\Delta u(\varepsilon,x)=0,\quad x\in\Omega_\varepsilon;\quad \mu\frac{\partial u}{\partial n}(\varepsilon,x)=q(x),\quad x\in\partial\Omega \tag{1.1}$$
$$\mu\frac{\partial u}{\partial x_2}(\varepsilon,x_1,\varepsilon\pm0)=\mp p_M^{\pm}(x_1),\quad a<x_1<b$$
$$\mu\frac{\partial u}{\partial x_2}(\varepsilon,x_1,\pm0)=\mp p_N^{\pm}(x_1),\quad -1<x_1<1$$

where u is the deplanation, μ is the shear modulus, n is the unit external normal vector to $\partial\Omega$, and q and $p_M^{\pm}$, $p_N^{\pm}$ are smooth external loads applied to the contour of $\partial\Omega$ and the edges $M_\varepsilon^{\pm}$, $N^{\pm}$ of the slits M_ε, N, respectively. We assume that the forces on the boundary of $\partial\Omega_\varepsilon$ are selfequilibrated, i.e, the following condition is satisfied:

$$\int_{\partial\Omega} q(x)\,dl+\sum_{\pm}\left\{\int_a^b p_M^{\pm}(x_1)\,dx_1+\int_{-1}^{1} p_N^{\pm}(x_1)\,dx_1\right\}=0 \tag{1.2}$$

for the solvability of the boundary value problem (1.1) in the space $W_2^1(\Omega_\varepsilon)$ (or in the class of bounded functions).

Let r_b, θ_b be polar coordinates with centre at the vertex (b,ε) of the crack M_ε such that the edges $M_\varepsilon^{\pm}$ are given by the relations $\theta_b=\pm\pi$. The representation

$$u(\varepsilon,x)=\text{const}+K_b(\varepsilon)\mu^{-1}(^1/_2 r_b/\pi)^{1/2}\sin{}^1/_2\theta_b+O(r_b|\ln r_b|) \tag{1.3}$$

holds for the solution u of problem (1.1) in a small neighbourhood of the point (b,ε) where $K_b(\varepsilon)$ is the stress intensity factor /4/. Analogous formulas also hold near the ends (a,ε) and $(\pm1,0)$ of the slits M_ε and N. We denote the appropriate intensity factors by $K_a(\varepsilon)$ and $K_{\pm}(\varepsilon)$.

A method of solving this problem and a broader class of problems in ideal fluid flow is developed in /1/. It utilized conformal mapping and enables the problem to be reduced to an evaluation of quadratures. The purpose of this paper is to construct an asymptotic expansion in the parameter ε. Taking account of the smallness of ε, the asymptotic solution of problem (1.1) is expressed in terms of the solution of a simpler problem in the domain Ω with one slit N (which can be solved, in turn, by using the method described in /1/). Approximate formulas for the intensity coefficients that clarify their qualitative dependence on the small spacing between the cracks are obtained as a result. In the case of canonical domains, when the limit problem has a solution in analytic form, the relationships obtained acquire an especially explicit form.

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2. Asymptotic form of the solution in a narrow strip between cracks. We consider (1.1) and the boundary conditions (1.3), (1.4) as a boundary value problem in a thin domain $\Pi_\varepsilon = \{x\colon 0 < x_2 < \varepsilon,\ a < x_1 < b\}$. Following /5-7/, we will seek the asymptotic expression of the function u in the form of the sum

$$u(\varepsilon, x) \sim \varepsilon^{-1} w_0(x_1) + w_1(x_1) + \varepsilon W_0(x_1, \varepsilon^{-1} x_2) \equiv W(\varepsilon, x) \tag{2.1}$$

Substituting (2.1) into the equation and the last two conditions in (1.1) and equating coefficients of ε^{-1} and ε_0, we obtain the relationships

$$\mu \frac{\partial^2 W_0}{\partial \eta^2}(x_1, \eta) + \mu \frac{d^2 w_0}{d x_1^2}(x_1) = 0, \quad \eta \in (0, 1) \tag{2.2}$$
$$\mu \frac{\partial W_0}{\partial \eta}(x_1, 1) = p_M^-(x_1), \quad \mu \frac{\partial W_0}{\partial \eta}(x_1, 0) = -p_N^+(x_1); \quad \eta = \frac{x_2}{\varepsilon}$$

If (2.2) is considered as a boundary value problem in the function W_0 (with the parameter $x_1 \in (a, b)$), then the equation

$$\mu \frac{d^2 w_0}{d x_1^2}(x_1) = -p_M^-(x_1) - p_N^+(x_1), \quad x_1 \in (a, b) \tag{2.3}$$

which must be considered as an equation for the unknown function w_0, is the condition for its solvability. The necessary boundary conditions for (2.3) will be determined in Sect.5 when studying boundary layers near the points (a, ε) and (b, ε).

The equation for the function w_1 in (2.1) has the same form as (2.3), and is found by using the same reasoning (see Sect.5 of /7/, say). However, the function w_1 is not needed to construct the principal term of the asymptotic expression of u. We merely note that the equation mentioned has a zero right-hand side, i.e., w_1 is a linear function.

3. The asymptotic form of the solution far from Π_ε. If we set $\varepsilon = 0$, then the domain Ω_ε is transformed into the domain Ω_0 with a single crack N. The boundary value problem (1.1) hance transforms into the following

$$\mu \Delta v_0(x) = 0, \quad x \in \Omega_0; \quad \mu \frac{\partial v_0}{\partial n}(x) = q(x), \quad x \in \partial\Omega \tag{3.1}$$
$$\mu \frac{\partial v_0}{\partial x_2}(x_1, +0) = -p_N^+(x_1), \quad x_1 \in (1, a) \cup (b, 1)$$
$$\mu \frac{\partial v_0}{\partial x_2}(x_1, +0) = -p_M^+(x_1), \quad x_1 \in (a, b);$$
$$\mu \frac{\partial v_0}{\partial x_1}(x_1, -0) = p_N^-(x_1), \quad x_1 \in (-1, 1)$$

Problem (3.1) cannot have a bounded solution since by virtue of (1.2)

$$\mu \int_{\partial\Omega_0} \frac{\partial v_0}{\partial n}(x)\, ds = -I, \quad I = \int^{b} (p_N^+(x_1) + p_M^-(x_1))\, dx_1 \tag{3.2}$$

Consequently, it is necessary to expand the class of functions allowable as solutions. Namely, we extract the points $(a, +0)$ and $(b, +0)$ that are images of the tips of the crack M_ε, and we permit the functions u to have logarithmic singularities at these points. Then the boundary value problem becomes solvable; however, its solution will be determined to the accuracy of a linear combination of two functions satisfying the homogeneous problem. The first is identically equal to one, while the second agrees with the Neumann function G whose poles are at the points $(a, +0)$ and $(b, +0)$. We recall that the function G satisfies the relationships

$$\Delta G(x) = 0, \quad x \in \Omega_0; \quad \frac{\partial G}{\partial n}(x) = 0, \quad x \in \partial\Omega_0 \tag{3.3}$$

$$G(x) = -\pi^{-1} \ln r_b + G_b + O(r_b), \; x_2 > 0, \; r_b \to 0 \tag{3.4}$$

$$G(x) = \pi^{-1} \ln r_a + G_a + O(r_a), \; x_2 > 0, \; r_a \to 0 \tag{3.5}$$

where G_a and G_b are certain constants.

Thus, we select the linear combination

$$u(\varepsilon, x) \sim c_v + V_0(x) + A_0(\varepsilon) G(x) \equiv V(\varepsilon, x) \tag{3.6}$$

as the asymptotic expression of the function u (as a solution of problem (3.1)), where c_v is an arbitrary constant (rigid displacement), the quantity $A_0(\varepsilon)$ is to be determined, and V_0 is a function bounded outside any neighbourhood of the point $(b, 0)$ and satisfying (3.1) and subject to the relationship

$$V_0(x) = \pi^{-1} I \mu^{-1} \ln r_b + V_b^0 + O(r_b |\ln r_b|), \quad x_2 > 0, \quad r_b \to 0 \tag{3.7}$$

4. Boundary layers near the tips of the crack M_ε. A formal asymptotic expression of the functions u inside and outside Π_ε was found in Sects.2 and 3. In order to combine these representations, and therefore, eliminate the arbitrariness in the selection of certain constants, we will study the behaviour of the solution of problem (1.1) in the neighbourhoods of the points (a, ε) and (b, ε). As usual, a boundary layer originates in these zones. By virtue of the symmetry of the problem, it is sufficient to consider just one tip of the crack M_ε, the point (b, ε), to be specific. We make the change of coordinates $x \to \xi = \varepsilon^{-1}(x_1 - b, x_2)$ a stretching of the domain Ω_ε ε^{-1} times relative to the point mentioned. Transferring to $\varepsilon = 0$ and confining ourselves to a consideration of the equations for $\xi_2 \geqslant 0$, we obtain the boundary value problem

$$\mu\Delta z(\xi) = 0, \quad \xi \in \Xi; \quad \mu \frac{\partial z}{\partial \xi_2}(\xi) = 0, \quad \xi \in \partial\Xi \tag{4.1}$$

where $\Xi = \mathbf{R}_+^2 \setminus \{\xi \in \mathbf{R}^2\colon \xi_2 = 1, \ \xi_1 < 0\}$ is the upper half-plane with a cutout ray (Fig.2).

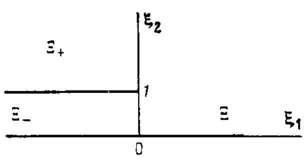

Fig.2

The domain Ξ has two "exists" at infinity: in the form of the angle Ξ_+ and the half-pole Ξ_-. We will list the solutions of problem (4.1) that have not more than polynomial growth in Ξ and allow the estimate $O(|\ln|\xi||^m)$ in Ξ_+. One such solution ζ_0 is obvious: $\zeta_0(\xi) = 1$. From the results of /8, 9/ it follows that every solution ζ_1 possessing the properties mentioned will allow of the representation

$$\begin{aligned} &\zeta_1(\xi) = c_1\xi_1 + c_2 + O(\exp(\pi\xi_1)) \quad \text{as} \quad \xi_1 \to -\infty \quad \text{in} \quad \Xi_- \\ &\zeta_1(\xi) = c_3 \ln|\xi| + c_4 + O(|\xi|^{-1}|\ln|\xi||) \quad \text{as} \quad |\xi| \to \\ &\to \infty \quad \text{in} \quad \Xi_+ \end{aligned} \tag{4.2}$$

where c_j are certain constants. We substitute ζ_1 and ζ_0 into the Green's formula for the domain $\Xi_R = \{\xi \in \Xi\colon |\xi| < R$ for $\xi \in \Xi_+$, and $\xi_1 > -R$ for $\xi \in \Xi_-\}$, where R is a large positive number. We have

$$\begin{aligned} 0 = &\int\limits_{\Xi_R} (\zeta_1(\xi)\Delta\zeta_0(\xi) - \zeta_0(\xi)\Delta\zeta_1(\xi))\,d\xi = \\ &\int\limits_{\partial\Xi_R} \left(\zeta_1(\xi)\frac{\partial\zeta_0}{\partial n}(\xi) - \zeta_0(\xi)\frac{\partial\zeta_1}{\partial n}(\xi)\right) dl \end{aligned} \tag{4.3}$$

where dl is an element of the length of the arc. The integrand in the last integral of (4.3) differs from zero only for the integral I_+ along the arc $\{\pi - \arcsin(R^{-1}) > \theta > 0, \ |\xi| = R\}$ and for the integral I_- along the segment $\{\xi_1 = -R, \ 0 < \xi_2 < 1\}$. Using (4.2), we find that

$$\begin{aligned} &I_+ = -\int\limits_0^{\pi - \arcsin(R^{-1})} (c_3 + O(R^{-1}\ln R))\,d\theta = -\pi c_3 + O(R^{-1}\ln R) \\ &I_- = \int\limits_0^1 (c_1 + O(\exp(-\pi R)))\,d\xi_2 = c_1 + O(\exp(-\pi R)) \end{aligned} \tag{4.4}$$

Therefore, passing to the limit as $R \to \infty$ and taking account of (4.4), we derive the equation $c_1 = \pi c_3$ from (4.3).

It only remains to note the following. If it is assumed that the solution z of problem (4.1) has the asymptotic expression (4.2) for $c_1 = c_3 = 0$, then it follows from Green's formula of the form (4.3) for ζ_1 and z that $c_2 = c_4$. Moreover, the solution that vanishes at infinity and corresponds to zero constants c_j in (4.2) possesses a finite Dirichlet integral, and therefore, is trivial.

Thus, all the linearly independent solutions sought for problem (4.1) that have the mentioned growth at infinity are exhausted by two: ζ_0 and ζ_1. The function ζ_1 is determined by using conformal mapping of the half-plane into the domain Ξ:

$$\eta_1 + i\eta_2 \to \xi_1 + i\xi_2 = \pi^{-1}(e^{-1}(\eta_1 + i\eta_2) + \ln(\eta_1 + i\eta_2)) \tag{4.5}$$

Namely, if $\gamma\colon \xi_1 + i\xi_2 \to \eta_1 + i\eta_2$ is a reciprocal function to (4.5), then ζ_1 is given by the equation /10/

$$\zeta_1(\xi) = \ln|\gamma(\xi_1 + i\xi_2)| \tag{4.6}$$

Direct calculations result in the following values of the constants c_j in the asymptotic forms (4.2) of the function (4.6)

$$c_1 = \pi, \ c_2 = 0, \ c_3 = 1, \ c_4 = 1 + \ln\pi \tag{4.7}$$

Later a representation of the type (1.3) is needed for the function ζ_1 near the tips of the slit

$$\zeta_1(\xi) = 1 + (2\pi\rho)^{1/2} \sin {}^1/_2\varphi + O(\rho) \tag{4.8}$$

where ρ, φ are polar coordinates with centre $(0, 1)$ such that the slit edges are given by the relations $\varphi = \pm\pi$; here $\varphi = \theta + O(\varepsilon)$, $\rho = \varepsilon^{-1}(r + O(\varepsilon))$ (see (1.3)).

We note finally that by allowing a linear growth of the function z in the corner Ξ_+ we obtain one more solution $\zeta_2(\xi) = \xi_1$ of the homogeneous boundary value problem (4.1) in addition to ζ_0 and ζ_1.

5. Combining asymptotic forms using boundary layers. We will seek an approximation of the solution u of problem (1.1) in small neighbourhoods of the points (a, ε) and (b, ε) in the form

$$u(\varepsilon, x) \sim c_a(\varepsilon) + B_a(\varepsilon)\zeta_1(\varepsilon^{-1}(a - x_1), \varepsilon^{-1}x_2) \equiv Z_a(\varepsilon, x) \tag{5.1}$$

$$u(\varepsilon, x) \sim c_b(\varepsilon) + B_b(\varepsilon)\zeta_1(\varepsilon^{-1}(x_1 - b), \varepsilon^{-1}x_2) \equiv Z_b(\varepsilon, x) \tag{5.2}$$

As mentioned earlier, the role of the boundary layers (5.1) and (5.2) is to combine the approximations to u. Using the method of combinable asymptotic expansions (/11-13/, etc.), we find the quantities $c_a(\varepsilon), c_b(\varepsilon), B_a(\varepsilon), B_b(\varepsilon)$ and $A_0(\varepsilon)$ in (5.1), (5.2) and (3.6) from the condition that the asymptotic expansions (2.1) and (3.6) should be in agreement in common intermediate zones.

If the point x is such that $0 < x_2 < \varepsilon, x_1 \sim b - \sqrt{\varepsilon}$, then

$$W(\varepsilon, x) = \varepsilon^{-1}w_0(0) + w_1(0) + \frac{x_1 - b}{\varepsilon}\frac{dw_0}{dx_1}(b) + \frac{(x_1 - b)^2}{2\varepsilon}\frac{d^2w_0}{dx_1^2}(b) + O(\sqrt{\varepsilon})$$

or in ξ coordinates (see Sect.4)

$$W(\varepsilon, (b, 0) + \varepsilon\xi) = \varepsilon^{-1}w_0(0) + w_1(0) + \xi_1\frac{dw_0}{dx_1}(b) + \frac{\varepsilon\xi_1^2}{2}\frac{d^2w_0}{dx_1^2}(b) + O(\sqrt{\varepsilon}) \tag{5.3}$$

Using (4.2) and (4.7), we find that

$$Z_b(\varepsilon, x) \sim c_b(\varepsilon) + B_b(\varepsilon)\pi\xi_1 + O(\exp(\pi\xi_1)) \tag{5.4}$$

Equating (5.3) and (5.4), we deduce the relations

$$c_b(\varepsilon) = \varepsilon^{-1}w_0(0) + w_1(0), \quad B_b(\varepsilon) \equiv B_b = \pi^{-1}\frac{dw_0}{\partial x_1}(b) \tag{5.5}$$

If $x_2 > \varepsilon$ and $r_b \sim \sqrt{\varepsilon}$, then according to (3.4)-(3.7) we have

$$V(\varepsilon, x) = c_v + \pi^{-1}I\mu^{-1}\ln r_b + V_b^0 + A_0(\varepsilon)(-\pi^{-1}\ln r_b + G_b) + O(\sqrt{\varepsilon}) \tag{5.6}$$

Moreover, in the same zone the formula

$$Z_b(\varepsilon, x) = c_b(\varepsilon) + B_b(\ln(r_b\varepsilon^{-1}) + \ln\pi + 1) + O(\sqrt{\varepsilon}) \tag{5.7}$$

follows from (4.2) and (4.7).

The relations

$$\mu^{-1}I - A_0(\varepsilon) = \pi B_b, \quad c_v + V_b^0 + A_0(\varepsilon)G_b = c_b(\varepsilon) + B_b(\ln(\varepsilon^{-1}\pi) + 1) \tag{5.8}$$

result from the assumption about the coincidence of the asymptotic expressions (5.6) and (5.7).

In considering the boundary layer Z_a the relationships

$$c_a(\varepsilon) = \varepsilon^{-1}w_0(a) + w_1(a), \quad B_a(\varepsilon) \equiv B_a = -\pi^{-1}\frac{dw_0}{dx_1}(a) \tag{5.9}$$

$$A_0(\varepsilon) = \pi B_a, \quad c_v + V_a^0 + A_0(\varepsilon)G_a = c_a(\varepsilon) + B_a(\ln(\varepsilon^{-1}\pi) + 1)$$

are deduced in exactly the same way.

Solving the system of algebraic Eqs.(5.5), (5.8), (5.9), we find the missing boundary conditions for the function w_0 satisfying (2.3)

$$w_0(b) - w_0(a) = 0, \quad \frac{dw_0}{dx_1}(b) - \frac{dw_0}{dx_1}(a) = I\mu^{-1} \tag{5.10}$$

the expressions for the unknown constant

$$A_0(\varepsilon) \equiv A_0 = -\frac{1}{\pi}\frac{dw_0}{dx_1}(a), \quad B_a = -\frac{1}{\pi}\frac{dw_0}{dx_1}(a), \quad B_b = \frac{1}{\pi}\frac{dw_0}{dx_1}(b) \tag{5.11}$$

and also one of the boundary conditions for the function w_1

$$w_1(b) - w_1(a) = V_b^0 - V_a^0 - \frac{1}{\pi}\frac{dw_0}{dx_1}(a)(G_b - G_a) + \frac{1}{\pi}\left(1 + \ln\frac{\pi}{\varepsilon}\right)\left(\frac{dw_0}{dx_1}(b) + \frac{dw_0}{dx_1}(a)\right)$$

The second boundary condition and, as remarked earlier, the equation for w_1 are determined when constructing the next terms of the asymptotic forms.

We note that problem (2.3), (5.10) is solvable (according to the definition (3.2) of the quantity I and the assumption (1.2) about the selfequilibration of the load).

6. Foundation for the asymptotic expansion. We assume the solutions u, V and G of problems (1.1), (3.1), (3.7) and (3.3)-(3.5) are normalized by the conditions

$$\int_{\partial\Omega} u(\varepsilon, x)\,dx = \int_{\partial\Omega} V(\varepsilon, x)\,dx = \int_{\partial\Omega} G(x)\,dx = 0 \tag{6.1}$$

Let χ be the cutoff function from $C_0^\infty(\mathbf{R}^1)$ such that $\chi(t) = 0$ for $|t| \geqslant 1$ and $\chi(t) = 1$ for $|t| < {}^1/_2$ while $\chi(\varepsilon, x) = (1 - \chi(r_a\varepsilon^{-1/2}))(1 - \chi(r_b\varepsilon^{-1/2}))$. We introduce the function

$$\begin{aligned} &U(\varepsilon, x) = V(\varepsilon, x), \quad x \in \Omega_\varepsilon, \quad x_2 < 0 \\ &U(\varepsilon, x) = \chi(\varepsilon, x)V(\varepsilon, x) + \chi(r_a\varepsilon^{-1/2})Z_a(\varepsilon, x) + \\ &\quad \chi(r_b\varepsilon^{-1/2})Z_b(\varepsilon, x), \quad x \in \Omega_\varepsilon \setminus \overline{\Pi}_\varepsilon, \quad x_2 > 0 \\ &U(\varepsilon, x) = (1 - \chi((x_1 - a)\varepsilon^{-1/2})(1 - \chi((x_1 - b)\varepsilon^{-1/2}W(\varepsilon, x) + \\ &\quad \chi((x_1 - a)\varepsilon^{-1/2})Z_a(\varepsilon, x) + \chi((x_1 - b)\varepsilon^{-1/2})Z_b(\varepsilon, x), \quad x \in \Pi_\varepsilon \end{aligned} \tag{6.2}$$

(see (2.1), (3.6), (5.1), (5.2)). It is clear that U satisfies the boundary conditions from (1.1) on N, M_ε^- and $\partial\Omega$ but leaves a residual $\Phi(\varepsilon, x_1)$ on M_ε^+ which allows of the estimate

$$|\Phi(\varepsilon, x_1)| \leqslant c_1\varepsilon\,[(x_1 - a + \varepsilon)(b - x_1 + \varepsilon)]^{1/2} \tag{6.3}$$

Since the functions V_0, Z_a and Z_b are harmonic, residual $\Psi(\varepsilon, x)$ of the approximation U in (1.1) is concentrated in the union of Π_ε and the $\sqrt{\varepsilon}$-neighbourhoods of the points $(b, +0)$ and $(a, +0)$. By virtue of the agreement of the asymptotic forms V and Z, W and Z near these points, the following relations hold:

$$\begin{aligned} &|\Psi(\varepsilon, x)| \leqslant c_2(r_a + \varepsilon)^{-1}, \quad r_a < \sqrt{\varepsilon}, \quad x \overline{\in} \Pi_\varepsilon \\ &|\Psi(\varepsilon, x)| \leqslant c_3(r_b + \varepsilon)^{-1}, \quad r_b < \sqrt{\varepsilon}, \quad x \overline{\in} \Pi_\varepsilon \\ &|\Psi(\varepsilon, x)| \leqslant c_4\varepsilon(\varepsilon + (x_1 - a)(b - x_1))^{-1}, \quad x \in \Pi_\varepsilon \end{aligned} \tag{6.4}$$

The estimate

$$\begin{aligned} &\int_{\Omega_\varepsilon} \{|\nabla u - \nabla U|^2 + d^2|u - U|^2\}\,dx \leqslant c_5\left\{\int_{\Omega_\varepsilon} d^{-2}\Psi^2\,dx + \int_{M_\varepsilon^+} \Phi^2\,dx_1\right\} \\ &d(x) = r_a^{-1}(|\ln r_a| + 1)^{-1} + r_b^{-1}(|\ln r_b| + 1)^{-1} \end{aligned} \tag{6.5}$$

results (according to (6.1)) from the one-dimensional Hardy inequality and the Poincaré-Friedrichs inequalities.

By virtue of (6.3) and (6.4), the integral over Ω_ε from the right side of (6.5) does not exceed $c_6\varepsilon|\ln\varepsilon|^2$ and the integral over M_ε^+ does not exceed $c_7\varepsilon$. Hence

$$\|u - U;\ W_2^1(\Omega_\varepsilon)\| \leqslant c_8\sqrt{\varepsilon}\,|\ln\varepsilon| \tag{6.6}$$

To simplify the discussion we will confine oursevles here to the deduction of just a rough estimation of the closeness of the constructed approximation (6.2) to the solution of the problem (1.1). The inequality (6.6) can be refined by using the method in /14/. In particular, the relationships

$$\begin{aligned} &u(\varepsilon, x) = V(\varepsilon, x) + O(\varepsilon|\ln\varepsilon|) \quad \text{in} \quad \Omega_\varepsilon \setminus \Pi_\varepsilon \setminus D_\delta \\ &u(\varepsilon, x) = Z(\varepsilon, x) + O(\varepsilon|\ln\varepsilon|) \quad \text{in} \quad D_\varepsilon;\ D_\delta = \{x \in \Omega_\varepsilon : \min\{r_a, r_b\} < \delta\} \end{aligned} \tag{6.7}$$

hold, where δ is a fixed positive number. We note that the operator $\min\{\varepsilon, r_a, r_b\}\,\partial/\partial x_j$ can be applied to the left-hand sides of (6.7) without degrading the estimate of the residue.

7. Asymptotic expression for the intensity factors. Asymptotic formulas for the stress intensity factors at the tips of the cracks M_ε and N are a result of the representations (6.7) of problem (1.1) (see (1.3)). We introduce the coefficients $k_V^\pm$ and $k_G^\pm$ into the expansions

$$V_0(x)=\mathrm{const}_\pm+\mu^{-1}k_V^\pm\left({}^1/_2\frac{r_\pm}{\pi}\right)^{1/2}\sin{}^1/_2\theta_\pm+O(r_\pm|\ln r_\pm|)\tag{7.1}$$

$$G(x)=\mathrm{const}^\pm+\mu^{-1}k_G^\pm\left({}^1/_2\frac{r_\pm}{\pi}\right)^{1/2}\sin{}^1/_2\theta_\pm+O(r_\pm)\quad \text{as}\quad r_\pm\to 0\tag{7.2}$$

of the harmonic functions V_0 and G (see (3.7) and (3.3)-(3.5)). Here $r_\pm, \theta_\pm$ are polar coordinates with centres $(\pm 1,0)$ such that the edges $N^\pm$ of the slit N are given by the equations $\theta_\pm=\pi$ and $\theta_\pm=-\pi$.

Comparing formulas (1.3), (3.6), (7.1), (7.2) and (5.11), we find that

$$K_\pm(\varepsilon)=k_V^\mp-\pi^{-1}\frac{dw_0}{dx_1}(a)\,k_G^\mp+O(\varepsilon|\ln\varepsilon|)\tag{7.3}$$

Similarly we deduce the following relationships from (5.1), (5.2) and (4.8)

$$K_a(\varepsilon)=2\mu\varepsilon^{-1/2}\frac{dw_0}{dx_1}(a)+O(\sqrt{\varepsilon}|\ln\varepsilon|)\tag{7.4}$$

$$K_b(\varepsilon)=-2\mu\varepsilon^{-1/2}\frac{dw_0}{dx_1}(b)+O(\sqrt{\varepsilon}|\ln\varepsilon|)$$

The solution w_0 of the oridinary differential Eqs(2.3), (5.10) is contained in the asymptotic formulas (7.3), (7.4). If $w_0=\mathrm{const}$, then the representations (7.4) become of little interest. The latter holds, say, if the crack edges are stress-free, i.e., $p_M{}^\pm=p_N{}^\pm=0$ in (1.1). We consider the situation mentioned by constructing the second term of the asymptotic form (its proof is carried out exactly as in Sect.6 and is omitted here).

Far from Π_ε we seek the asymptotic form u in the form

$$u(\varepsilon,x)\sim v_0(x)+\varepsilon v_1(x)+\varepsilon A_1G(x)\tag{7.5}$$

where A_1 is a certain constant and the function v_0 satisfies relationship (3.1) and the boundary condition

$$\mu\frac{\partial v_0}{\partial x_2}(x)=0,\quad x\in N^\pm\tag{7.6}$$

v_1 is the solution of the boundary value problem

$$\Delta v_1(x)=0,\quad x\in\Omega_0;\quad \frac{\partial v_1}{\partial n}(x)=0,\quad x\in\partial\Omega\tag{7.7}$$

$$\frac{\partial v_1}{\partial x_2}(x_1,+0)=\frac{\partial^2 v_0}{\partial x_1^2}(x_1,+0),\quad x_1\in(a,b)\tag{7.8}$$

$$\frac{\partial v_1}{\partial x_2}(x)=0,\quad x\in N^-\cup(N^+\setminus M_0{}^+)$$

Let us clarify the reasons for this choice of the right-hand sides of the boundary conditions (7.8). Expanding the function on the right in (7.5) in a Maclauren series, we find that for $x\in M_\varepsilon{}^+$ (or $x_1\in(a,b)$, $x_2=\varepsilon+0$)

$$\frac{\partial u}{\partial x_2}(\varepsilon,x_1,\varepsilon+0)\sim\frac{\partial v_0}{\partial x_2}(x_1,+0)+\varepsilon\left(\frac{\partial v_1}{\partial x_2}(x_1,+0)+A_1\frac{\partial G}{\partial x_2}(x_1,+0)+\frac{\partial^2 v_0}{\partial x_2^2}(x_1,+0)\right)+O(\varepsilon^2)\tag{7.9}$$

Since v_0 and G satisfy the homogeneous Neumann conditions on N^+, the coefficient of ε from (7.9) vanishes if

$$\frac{\partial v_1}{\partial x_2}(x_2,+0)\equiv-\frac{\partial^2 v_0}{\partial x_2^2}(x_1,+0)=\frac{\partial^2 v_0}{\partial x_1^2}(x_1,+0)-\Delta v_0(x_1,+0)=\frac{\partial^2 v_0}{\partial x_1^2}(x_1,+0)$$

According to Sect.3, a solution of the problem (7.7), (7.8) exists that is bounded outside any neighbourhood of the point $(b,+0)$ and allows of the representation (compare with (3.8))

$$v_1(x)=\pi^{-1}I_1\ln r_b+V_b{}^1+O(r_b|\ln r_b|)\quad \text{as}\quad r_b\to 0,\ x_2>0\tag{7.10}$$

$$I_1=-\int_{\partial\Omega_0}\frac{\partial v_1}{\partial n}(x)\,ds=-\int_a^b\frac{\partial^2 v_0}{\partial x_1^2}(x_1,+0)\,dx_1=\frac{\partial v_0}{\partial x_1}(a,+0)-\frac{\partial v_0}{\partial x_1}(b,+0)\tag{7.11}$$

Inside Π_ε the solution $u(\varepsilon, x)$ is approximated by the quantity $w_1(x_1)$ which is a linear function $\alpha x_1 + \beta$ (see Sect.2). The boundary layers (5.1) and (5.2) have the form

$$Z_a(\varepsilon, x) = c_a^0 + \varepsilon\left(c_a^1 + B_a\zeta_1\left(\frac{a-x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) + D_a\zeta_2\left(\frac{a-x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)\right) \quad (7.12)$$

$$Z_b(\varepsilon, x) = c_b^0 + \varepsilon\left(c_b^1 + B_b\zeta_1\left(\frac{x_1-b}{\varepsilon}, \frac{x_2}{\varepsilon}\right) + D_b\zeta_2\left(\frac{x_1-b}{\varepsilon}, \frac{x_2}{\varepsilon}\right)\right) \quad (7.13)$$

Performing the combination, we obtain algebraic equations as in Sect.5

$$\alpha b + \beta = v_0(b, +0),\ \alpha a + \beta = v_0(a, +0),\ \pi B_b = I_1 - A_1 \quad (7.14)$$
$$\pi B_a = A_1,\quad \alpha = \pi B_b + D_b = -\pi B_a - D_a,\quad D_b = \frac{\partial v_0}{\partial x_1}(b, +0),$$
$$D_a = -\frac{\partial v_0}{\partial x_1}(a, +0)$$

Solving the overdefined but solvable system (7.14), by virtue of (7.11), we find that

$$A_1 = \pi B_a = I_1 - \pi B_b, \alpha = (v_0(b, +0) - v_0(a, +0))(b-a)^{-1} \quad (7.15)$$
$$B_b = \frac{1}{\pi}\left(\alpha - \frac{\partial v_0}{\partial x_1}(b, +0)\right),\ B_a = \frac{1}{\pi}\left(\frac{\partial v_0}{\partial x_1}(a, +0) - \alpha\right)$$

Therefore, the following asymptotic formulas for the intensity factors result from (7.5), (7.12), (7.13) and (7.15) in the solution of problem (1.1) for $p_M^\pm = p_N^\pm = 0$

$$K_\pm(\varepsilon) = k_V^\pm + \varepsilon\left\{k_{1,v}^\pm + k_G^\pm\left(\frac{\partial v_0}{\partial x_1}(a, +0) - \frac{v_0(b, +0) - v_0(a, +0)}{b-a}\right)\right\} + O(\varepsilon^2|\ln\varepsilon|) \quad (7.16)$$

$$K_b(\varepsilon) = 2\mu\sqrt{\varepsilon}\left\{\frac{v_0(b, +0) - v_0(a, +0)}{b-a} - \frac{\partial v_0}{\partial x_1}(b, +0)\right\} + O(\varepsilon^{3/2}|\ln\varepsilon|) \quad (7.17)$$

where $k_{1,v}^\pm$ are coefficients in representations of the type (7.1) and (7.2) for the solution v_1 of problem (7.7), (7.8). (We note that the asymptotic form $K_a(\varepsilon)$ has the form (7.17) where the opposite sign should be taken and a and b interchanged).

When $\Omega = \mathbf{R}^2$ formulas (7.16) and (7.17) take a more specific form. In particular, for functions u harmonic in $\Omega_\varepsilon = \mathbf{R}^2 \setminus (N \cup M_\varepsilon)$ and subject to the relations

$$\mu\frac{\partial u}{\partial n}(\varepsilon, x) = 0,\quad x \in N \cup M_\varepsilon; \quad (7.18)$$
$$\mu\frac{\partial u}{\partial x_2}(\varepsilon, x) = q_0 + o(1) \quad \text{as} \quad |x| \to +\infty$$

the intensity factors are calculated from the formulas

$$K_+(\varepsilon) = 2q_0\sqrt{\pi}\left\{1 + \frac{\varepsilon}{2\pi}\left[\frac{1}{4}\ln\frac{(1+b)(1-a)}{(1+a)(1-b)} + \frac{1}{2}\frac{b-a}{(1-a)(1-b)} + \sqrt{\frac{1+b}{1-b}}\left(\frac{b}{\sqrt{1-b^2}} - \frac{a}{\sqrt{1-a^2}}\right) + \left(\sqrt{\frac{1+b}{1-b}} - \sqrt{\frac{1+a}{1-a}}\right)\left(\frac{a+b}{\sqrt{1-a^2}+\sqrt{1-b^2}} - \frac{a}{\sqrt{1-a^2}}\right)\right]\right\} + O(\varepsilon^2|\ln\varepsilon|) \quad (7.19)$$

$$K_b(\varepsilon) = 2q_0\sqrt{\varepsilon}\frac{1}{\sqrt{1-b^2}}\frac{b-a}{1-ab+\sqrt{(1-a^2)(1-b^2)}} + O(\varepsilon^{3/2}|\ln\varepsilon|)$$

8. Cracks shifted relative to one another.

We will investigate a crack arrangement different from that studied in Sects.1-7. We retain the same notation as in Sect.1 for a, b, ε and Ω. We set $M_\varepsilon = \{x \in \mathbf{R}^2\colon x_2 = \varepsilon, -1 \leqslant x_1 \leqslant b\}$, $N_1 = \{x \in \mathbf{R}^2\colon x_2 = 0, a \leqslant x_1 \leqslant 1\}$, $\Omega_\varepsilon = \Omega \setminus N_1 \setminus M_\varepsilon$ (Fig.3). We consider (1.1) with zero Neumann data on $N_1 \cup M_\varepsilon$. The asymptotic form of the solution is constructed by the same scheme as before. The sole difference is in the definition of the function v_1.

In the case under consideration the residual $\partial^2 v_0/\partial x_1^2$ occurs in the set $M_\varepsilon^+ \cup \{x \in M_\varepsilon^-\colon -1 \leqslant x_1 \leqslant a\}$, and consequently, the right-hand sides of the boundary conditions of the type (7.8) have the inadmissible growth $O(r_-^{-3/2})$ as $r_- \to 0$. The reason for such growth is that the image $(-1,0)$ of the left tip of the crack M_ε in the limit problem is shifted relative to the initial position and the function v_0 does not satisfy the boundary conditions near $(-1,0)$. Consequently, it is necessary to change the form of the fundamental approximation to u by selecting

$$v^*(\varepsilon, x) = X_\delta(x)\, v_0(x_1, x_2+\varepsilon) + (1 - X_\delta(x))\, v_0(x) \qquad (8.1)$$
$$X_\delta(x) = \chi(\delta^{-1}(x_1+1))\, \chi(\delta^{-1}x_2)$$

as its function, where χ is the cutoff from Sect.6, and δ is so small a number that the support X_δ intersects neither $\partial\Omega$ nor N_1. We note that outside the neighbourhood of the point $(-1, \varepsilon)$ the function (8.1) is expanded in a series of non-negative integer powers of ε; this expansion is obtained after application of Taylor's formula to the first term on the right in (8.1).

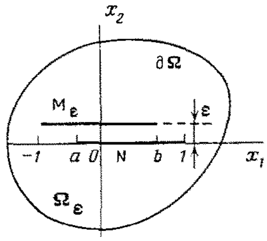

Fig.3

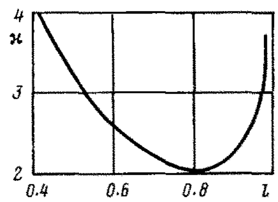

Fig.4

The principal term of the residual v^* in the boundary condition (1.1) has the form $\varepsilon\mu\varphi_1^\pm$, where

$$\varphi_1^+(x_1) = \chi(\delta^{-1}(x_1+1))\, \frac{\partial^2 v_0}{\partial x_1^2}(x_1, +0), \quad x_1 \in (-1, b)$$
$$\varphi_1^+(x_1) = 0, \quad x_1 \in (b, 1); \quad \varphi_1^-(x_1) = 0, \quad x_1 \in (a, 1)$$
$$\varphi_1^-(x_1) = \chi(\delta^{-1}(x_1+1))\, \frac{\partial^2 v_0}{\partial x_1^2}(x_1, -0), \quad x_1 \in (-1, a)$$

Moreover, a residual appears in (1.1) whose principal term agrees with the quantity $\varepsilon\mu\psi_1(x)$, where

$$\psi_1(x) = -\frac{\partial v_0}{\partial x_2}(x)\, \Delta X_\delta(x) - 2\nabla \frac{\partial v_0}{\partial x_2}(x) \cdot \nabla X_\delta(x)$$

Since v_0 is a harmonic function, then

$$\int\limits_{\Omega_0} \psi_1(x)\, dx = \int\limits_{\Omega_0} \left(\frac{\partial X_\delta}{\partial x_2}(x)\, \frac{\partial^2 v_0}{\partial x_1^2}(x) - \frac{\partial X_\delta}{\partial x_1}(x)\, \frac{\partial^2 v_0}{\partial x_1 \partial x_2}(x) \right) dx =$$
$$\sum_\pm \mp \int\limits_{-1}^{a} \frac{\partial \chi}{\partial x_1}(\delta^{-1}(x_1+1))\, \frac{\partial v_0}{\partial x_1}(x_1, \pm 0)\, dx_1$$

and consequently, a solution v_1 of the boundary value problem exists

$$\Delta v_1(x) = \psi_1(x), \quad x \in \Omega_0 \qquad (8.2)$$
$$\frac{\partial v_1}{\partial x_2}(x) = \pm\varphi_1^\pm(x_1), \quad x \in N^\pm; \quad \frac{\partial v_1}{\partial n}(x) = 0, \quad x \in \partial\Omega$$

which allows of the representation (7.10), (7.11), where $(a, +0)$ should be replaced by $(a, -0)$.

The remaining reasoning for the construction of the asymptotic expression is exactly the same as in Sect.7. We consequently obtain the asymptotic formulas (7.16), (7.17) for the intensity factors, in which $(a, +0)$ must be replaced by $(a, -0)$ and $k_{I,v}^\pm$ should be understood to be the factors in the asymptotic forms of the solution of problem (8.2), while $k_G^\pm$ are the factors in the expansion (7.2) of the Neumann functions with poles $(a, -0)$ and $(b, +0)$ (compare with (3.4) and (3.5)). In the case of problem (1.1), (7.18) in a plane, formulas (7.19) become

$$K_+(\varepsilon) = 2q_0\sqrt{\pi}\left\{1 + \frac{\varepsilon}{2\pi}\left[\frac{1}{4}\ln\frac{(1+b)(1-a)}{(1+a)(1-b)} + \frac{a+3b-4ab}{2(1-a)(1-b)} - \right.\right.$$
$$\left.\left.\left(\sqrt{\frac{1+b}{1-b}} + \sqrt{\frac{1+a}{1-a}}\right)\frac{\sqrt{1-b^2}+\sqrt{1-a^2}}{b-a}\right]\right\} + O(\varepsilon^2|\ln\varepsilon|)$$
$$K_b(\varepsilon) = 2q_0\sqrt{\varepsilon}\left\{\frac{1-ab+\sqrt{(1-a^2)(1-b^2)}}{(b-a)\sqrt{1-b^2}}\right\} + O(\varepsilon^{3/2}|\ln\varepsilon|)$$

Finally, we formulate one more result. We consider the crack arrangement shown in Fig.3 in the case when the cracks have the same unit length while the dimensions of the rectangle Π_ε between the cracks are $l \times \varepsilon$; $l \in (0, 1)$. The relationship

$$K(\varepsilon, l) = 2q_0 \sqrt{\varepsilon}\varkappa(l) + O(\varepsilon^{3/2} |\ln \varepsilon|)$$
$$\varkappa(l) = \frac{2\sqrt{1-l}}{l} + \frac{l}{2\sqrt{1-l}} = \frac{4 - 4l + l^2}{2l\sqrt{1-l}}$$

holds for the intensity factor $K(\varepsilon, l)$ at the right tip of the crack M_ε for problem (1.1), (7.18).

As $l \to 0$ and $l \to 1$ the quantity $\varkappa(l)$ tends to ∞; for values of l and $1 - l$ close to ε, the formula for $K(\varepsilon, l)$ loses accuracy because of the breakdown of the assumption about the smallness of ε (see Sect.1). A minimum of the function $\varkappa$ (Fig.4) and the stress intensity factor are achieved at the point $l_* = 2(\sqrt{2} - 1)$.

REFERENCES

1. CHEREPANOV G.P., Ideal fluid flows with free surfaces in doubly- and triply-connected domains. PMM, 27, 4, 1963.
2. PANASYUK V.V., SAVRUK M.P. and DATSYSHIN A.P., Stress Distribution Around Cracks in Plates and Shells, Naukova Dumka, Kiev, 1976.
3. BEREZHNITSKII L.T., PANASYUK V.V. and STASHUK N.G., Interaction between Rigid Linear Inclusions and Cracks in a Deformable Body. Naukova Dumka, Kiev, 1983.
4. CHEREPANOV G.P., Brittle Fracture Mechanics, Nauka, Moscow, 1974.
5. GOL'DENVEIZER A.L., Construction of an approximate plate bending theory by the method of asymptotic integration of the equations of elasticity theory, PMM, 26, 4, 1962.
6. ZINO I.E. and TROPP E.A., Asymptotic Methods in Problems of Heat Conduction and Thermo-elasticity Theories, Leningrad Univ. Press, Leningrad, 1978.
7. NAZAROV S.A., Introduction to Asymptotic Methods of Elasticity Theory, Leningrad Univ. Press, Leningrad, 1983.
8. KONDRAT'YEV V.A., Boundary value problems for elliptic equations in domains with conical or angular points, Trudy Moskov. Matem. Obsh., 16, 1967.
9. MAZ'YA V.G. and PLAMENEVSKII B.A., On coefficients in the asymptotic expressions of solutions of elliptic boundary value problems in domains with conical points, Math. Nachr., 76, 1977.
10. KANTOROVICH L.V. and KRYLOV V.I., Approximate Methods of Higher Analysis, Gostekhizdat, Moscow-Leningrad, 1950.
11. VAN DYKE M.D., Perturbation Methods in Fluid Mechanics, Mir, Moscow, 1967.
12. IL'IN A.M., A boundary value problem for a second-order elliptic equation in a domain with a slot. I. The two-dimensional case, Matem. Sb., 99, 4, 1976. II. A domain with a small hole, Matem. Sb., 103, 2, 1977.
13. FEDORYUK M.V., An asymptotic expression for the solution of the Dirichlet problem for the Laplace and Helmholtz equations on the exterior of a slender cylinder, Izv. Akad. Nauk SSSR, Ser. Mat., 45, 1, 1981.
14. MAZ'YA V.G., NAZAROV S.A. and PLAMENEVSKII B.A., The Asymptotic Form of the Solutions of Elliptic Boundary Value Problem for Singular Perturbed Domains. Tbil. Univ. Press, Tbilisi, 1981.

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